Math 217 Fall 2025

Quiz 21 – Solutions

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1. Complete* the partial sentences below into precise definitions for, or precise mathematical characterizations of, the italicized term:

   (a) Let $\mathfrak{B} = (v_1, \dots, v_d)$ be a basis for the vector space V. Let $v \in V$. The $\mathfrak{B}$-*coordinates* of v are ...

   Solution: The unique scalars $a_1, \dots, a_d \in \mathbb{R}$ such that

   $$v = a_1 v_1 + \cdots + a_d v_d.$$

   (b) Let $\mathfrak{B} = (v_1, \dots, v_d)$ be a basis for the vector space V. Let $v \in V$. The $\mathfrak{B}$-*coordinate column vector* of v is ...

   Solution: The column vector formed from the $\mathfrak{B}$-coordinates of v:

   $$[v]_{\mathfrak{B}} = \begin{bmatrix} a_1 \\ \vdots \\ a_d \end{bmatrix} \in \mathbb{R}^d, \quad \text{where } v = a_1 v_1 + \cdots + a_d v_d.$$

   (c) Let $T : V \to V$ be linear and $\mathfrak{B} = (v_1, \dots, v_n)$ an ordered basis of V. The *matrix of T with respect to $\mathfrak{B}$* is ...

   Solution: The $n \times n$ matrix $[T]_{\mathfrak{B}}$ whose j-th column is $[T(v_j)]_{\mathfrak{B}}$; i.e.

   $$[T]_{\mathfrak{B}} = \begin{bmatrix} | & & | \\ [T(v_1)]_{\mathfrak{B}} & \cdots & [T(v_n)]_{\mathfrak{B}} \\ | & & | \end{bmatrix}.$$

2. Suppose V is a vector space and $v_1, v_2, \dots, v_m$ are linearly independent vectors in V. Show that $v \in V \setminus \operatorname{Span}(v_1, \dots, v_m)$ if and only if the vectors $v, v_1, \dots, v_m$ are linearly independent.

   Solution: $(\Rightarrow)$ Suppose $v \notin \operatorname{Span}(v_1, \dots, v_m)$ and $a\,v + \sum_{i=1}^m b_i v_i = 0$. If $a \neq 0$, then

   $$v = -\frac{1}{a} \sum_{i=1}^m b_i v_i \in \operatorname{Span}(v_1, \dots, v_m),$$

*For full credit, please write out fully what you mean instead of using shorthand phrases.

a contradiction. Hence $a = 0$, and by independence of $v_1, \ldots, v_m$ we get $b_1 = \cdots = b_m = 0$.
($\Leftarrow$) If $v \in \operatorname{Span}(v_1, \ldots, v_m)$, then $v = \sum_{i=1}^m c_i v_i$ for some c_i, so

$$v - \sum_{i=1}^{m} c_i v_i = 0$$

is a nontrivial linear relation among $v, v_1, \ldots, v_m$, contradicting their independence. Thus $v \notin \operatorname{Span}(v_1, \ldots, v_m)$.

3. True or False. If you answer true, then state TRUE. If you answer false, then state FALSE. Justify your answer with either a short proof or an explicit counterexample.

(a) Suppose $V = \mathbb{R}^{2\times 2}$ with basis $\mathfrak{B} = \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right)$. Consider the linear transformation $T : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ that sends X to AX, where $A = \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}$. The matrix of T with respect to $\mathfrak{B}$ is

$$\begin{bmatrix} 2 & 0 & 0 & 0 \\ 3 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 3 & 2 \end{bmatrix}.$$

Solution: FALSE. Let $E_{11}, E_{12}, E_{21}, E_{22}$ denote the listed basis. Compute:

$$AE_{11} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} = 2E_{11},$$

$$AE_{12} = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} = 2E_{12},$$

$$AE_{21} = \begin{bmatrix} 3 & 0 \\ 2 & 0 \end{bmatrix} = 3E_{11} + 2E_{21},$$

$$AE_{22} = \begin{bmatrix} 0 & 3 \\ 0 & 2 \end{bmatrix} = 3E_{12} + 2E_{22}.$$

Thus the columns of $[T]_{\mathfrak{B}}$ are $\begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 0 \\ 2 \end{bmatrix}$, so

$$[T]_{\mathfrak{B}} = \begin{bmatrix} 2 & 0 & 3 & 0 \\ 0 & 2 & 0 & 3 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix},$$

which is not the given matrix.